Efficient purchaser incentive when dealing with suppliers implementing continuous improvement plans

Laoucine Kerbache and Christian van Delft

Mateus Hiro Nagata

March 26, 2025

Before Proceeding

Context

- Incentive schemes!
- Cost reduction can be implemented once in a year
- Random underlying process
- When?
- Purchaser knows stochastic process of cost reduction, supplier benefits from it

Model

Purchaser: knows potential cost reduction that occurs at each t, associated with the continuous improvement plan.

- c_t : Effective purchasing cost
- $\beta \in (0,1)$ Cost reduction factor
- $\xi_t \sim Bin(p) = 1_{\text{is there cost reduction factor at } t?}$
- $x_t = x_{t-1} + \tilde{\xi}_t$ cumulated number of unincorporated potential cost

Time periods buckets of T periods

$$t = mT + N, m \in \mathbb{N}, N = 1, \ldots, T$$

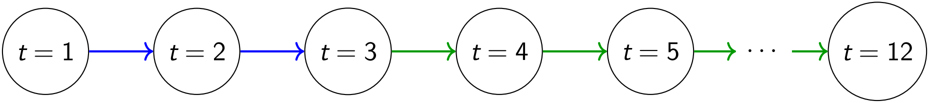
Suppose time bucket of size $T = 12$, then m represents years, N represents months in the year.

- At $t = mT + N, m \in \mathbb{N}$, purchaser observes $\tilde{\xi}_t$ (realization of the process ξ_t), and the current state $x_t = x_{t-1} + \tilde{\xi}_t$
- Choice

$$c_{t+1} = \begin{cases} c_t, \text{ if no change} \\ c_t \beta^{x_t} \text{ if change} \end{cases}$$

Effective cost from next period $t + 1 = mT + N + 1$ until $(m + 1)T$

Change at $t = 3$



$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=12$
$c_1 = 1$	$c_2 = c_1$	$c_3 = c_1$	$c_4 = c_3\beta^{x_3}$	$c_5 = c_3\beta^{x_3}$	$c_{12} = c_3\beta^{x_3}$
$\tilde{\xi}_1 = 0$	$\tilde{\xi}_2 = 1$	$\tilde{\xi}_3 = 1$	$\tilde{\xi}_4 = 0$	$\tilde{\xi}_5 = 1$	$\tilde{\xi}_{12} = 0$
x_1	$x_2 = x_1 + \tilde{\xi}_2$	$x_3 = x_2 + \tilde{\xi}_3$	$x_4 = x_3 + \tilde{\xi}_4$	$x_5 = x_4 + \tilde{\xi}_5$	$x_{12} = x_{12} + \tilde{\xi}_{12}$
		Change!			

Intuitions

- Cumulated available reduction x and time in the time bucket N matters
- x has a binomial distribution

$$\Xi^q := \sum_{l=1}^{q} \zeta_l$$

- ζ_l i.i.d. $\sim Bin(p)$

$$Prob[\Xi^q = k] = \frac{q!}{k!(q-k)!} p^k (1-p)^{q-k}, k = 0, \dots, q$$

- Average potential reduction over q periods

$$\hat{\beta}^q := E\left[\beta^{\Xi^q}\right] = \sum_{i=0}^{q} \left[\frac{q!}{i!(q-i)!} p^i (1-p)^{q-i} \beta^i\right]$$

Company Objective

minimize the **expected discounted unit purchasing cost over an infinite horizon** through purchase cost reductions

$$\Pi^* = \min \tilde{c}_1 E\left[\sum_{t=1}^{\infty} \alpha^t c_t \;\middle|\; c_1 = 1 \right]$$

- α : discount factor
- $\tilde{c}_1$: cost at the beginning of the first period

Markov Decision Process

Optimal Policy: Model Structure and Managerial Insight

Incentive Schemes and Purchaser Decision Process

Why Markov?

- Think of the choice in a yearly basis
- Cost, cumulated cost reduction
- No long-term dependency

- State (N, c, x), N: periods in current time bucket, c: cost at the beginning of current decision time period, x: cumulated available cost reductions.
- kth decision's time period:

$$t^k = m^k T + N^k$$

- $(N^k, c^k, x^k), c^k := c_{t^k}$: effective cost at period $t^k, x^k := x_{t^k}$

Transition cost

$\gamma(N, x, a)$: **Transition cost** at time N (in the time bucket), when cumulated available cost reductions x and action taken is:

$$\gamma(N, x, a) = \begin{cases} 1 \text{ if a} = 0 \\ \gamma(N, x, 1) = 1 + \left[\sum_{i=1}^{T-N} \alpha^i\right] \beta^x \text{ if } a = 1 \end{cases}$$

For instance, at $k = 1$, the **Discounted Transition Cost**:

$$\alpha^{t^1} c^1 \gamma(N^1, x^1, a^1)$$

and determines the **Next decision period**

$$t^2 = \zeta(t^1, a^1) = m^2 T + N^2$$

$$(m^2, N^2) = \begin{cases} (m^1, N^1 + 1) \text{ if } a^1 = 0, N^1 < t \\ (m^1 + 1, 1), \text{ otherwise} \end{cases}$$

Cost

$$c^2 := c_{t^2} = \begin{cases} c^1 \text{ if } a^1 = 0 \\ c^1 \beta^{x^1} \text{ if } a^1 = 1 \end{cases}$$

Cumulated number of potential cost reductions available at the second decision period is

$$x^2 := x_{t^2} = \begin{cases} x^1 + \tilde{\Xi}^1 \text{ if } a^1 = 0 \\ \tilde{\Xi}^{T+1-N^1} \text{ if } a^1 = 1 \end{cases}$$

If change

$t^1 = 1, (m^1 = 0, N^1 = 1)$

1. State observed $c^1 = 1, x^1 = \tilde{\xi}_1$
2. Action $a^1 = 1$ (change)
3. Induced discounted transaction cost

$$\alpha^{t^1} c^1 \gamma(N^1, x^1, a^1) = \alpha \left(1 + \left[\sum_{i=1}^{T-N_1} \alpha^i \right] \beta^{x^i} \right)$$

4. Determine the next decision period

$$t^2 = \zeta(t^1, a^1) = m^2 T + N^2$$

5. Cost $c^2 := c_{t^2} = c^1 \beta^{x^1}$
6. Cumulated number of potential cost reductions available at the second decision period

$$x^2 := x_{t^2} = \tilde{\Xi}^{T+1-N^1}$$

In general

$t^1 = 1, (m^1 = 0, N^1 = 1)$

1. State observed $c^1 = 1, x^1 = \tilde{\xi}_1$
2. Action a^1
3. Induced discounted transaction cost

$$\alpha^{t^1} c^1 \gamma(N^1, x^1, a^1)$$

4. Determine the next decision period

$$t^2 = \zeta(t^1, a^1) = m^2 T + N^2$$

5. Cost $c^2 := c_{t^2} = c^1$
6. Cumulated number of potential cost reductions available at the second decision period

$$x^2 := x_{t^2} = x^1 + \tilde{\Xi}^{T+1-N^1}$$

Markovian policy

Decision rule:

$$c^k = c^{k-1}\beta^{x^{k-1}\alpha^{k-1}} = c^1\beta^{\sum_{i=1}^{k} x^i a^i}$$

Policy $a^k = \pi(N^k, x^k)$.

Normalized expected discounted cost (given initial state $(j, x) \in \mathbb{N}^0 \times \mathbb{N}$

$$V_\pi(j, x) = E_\pi\left[\sum_{k=1}^{\infty} \alpha^{t^k} c^k \gamma(N^k, x^k, \pi(N^k, x^k)) \,\middle|\, t^1 = j, c^1 = 1, x^1 = x\right]$$

Optimal discounted cost (defined by optimal policy $\pi^* \; \forall(N, x)$)

$$V^*(N, x) = V_{\pi^*}(N, x) = \min_\pi\{V_\pi(N, x)\}$$

Optimal solution of the purchasing problem

$$\Pi^* = \tilde{c}_1 E_\Xi[V^*(1,\Xi)] = \tilde{c}_1[pV^*(1,1) + (1-p)V^*(1,0)]$$

Property (Markovian Property)

Given the stationarity and periodicity of the problem, for any Markovian policy $\pi(\cdot,\cdot), m \in \mathbb{N}, 1 \leq N \leq T$,

$$V_\pi(mT + N, x) = \alpha^{mT} V\pi(N, x).$$

Dynamic Programming formulation of the model

Local return function (immediate return)

$$h(N,x,a,V(\cdot,\cdot)) = \begin{cases} \alpha^N \gamma(N,x,0) + E_{\Xi}[V(N+1, x+\Xi)], & \text{if } a=0, \\ \alpha^N \gamma(N,x,1) + \alpha^T \beta^x E_{\Xi^{T-N+1}}[V(N+1, x+\Xi)], & \text{if } a=1. \end{cases}$$

Functional equations

$$V_\pi(N,x) = h(N,x,\pi(N,x),V_\pi(\cdot,\cdot))$$

$$V^*(N,x) = \min_{a\in\{0,1\}} h(N,x,a,V^*(\cdot,\cdot))$$

Markov Decision Process

Optimal Policy: Model Structure and Managerial Insight

Incentive Schemes and Purchaser Decision Process

- Difficult to solve analytically
- Consider extreme and simple cases

Property (Never too late)

For the optimal policy $\pi^*(\cdot,\cdot)$, *we have for any* $x \in \mathbb{N} \cup \{0\}$

$$\pi^*(T, x) = 1$$

Property (Optimal policy is threshold-like)

For the optimal policy $\pi^*(\cdot,\cdot) \forall N, 1 \leq N \leq T, \exists$ *a finite upper limit* $\bar{x}_N$ *such that for any* $x \in \mathbb{N}, x \geq \bar{x}_N$, *we have*

$$\pi^*(N, x) = 1$$

Property (Optimal policy is threshold-like (2))

For the optimal policy $\pi^*(\cdot,\cdot)$, *if a given pair* (N, x) *with* $x \in \mathbb{N}$ *and* $1 \leq N \leq T$, *we have* $\pi^*(N, x) = 1$, *then for any* $y \in \mathbb{N}$, *with* $y \geq x$, *we have* $\pi^*(N, y) = 1$

Lemma (Lower Bounds)

Policy $\underline{\pi}(\cdot,\cdot)$, *where we apply cost reduction after every period, creates an upper bound. This policy* $\underline{\pi}(\cdot,\cdot)$ *induces a lower bound cost.*

$$V_{\underline{\pi}}(N,x) = \alpha\frac{1-\alpha^N}{1-\alpha} + \frac{\alpha^{N+1}\beta^x}{1-\alpha p\beta}$$

and lower bound value Π^* *is given by*

$$\underline{\Pi} = \frac{\alpha\tilde{c}_1}{1-\alpha p\beta}$$

Lemma (Kth determinisitic change)

Change at the kth time period of each T-time bucket.

$$\bar{\pi}_k(x, N) = \begin{cases} 1, & \text{if } N = k \\ 0, & \text{otherwise} \end{cases}$$

Upper bound cost

$$V_{\pi_k}(N, x) = \begin{cases} \frac{\alpha^N(1-\alpha^{k-N})}{1-\alpha} + \left(\beta^x \hat{\beta}^{k-n} \frac{1-\alpha^T}{1-\alpha} \frac{\alpha^{k+1}}{1-\alpha^T \hat{\beta}^T}\right) & \text{if } N \le k \\ \frac{\alpha^N(1-\alpha^{T+k-N})}{1-\alpha} + \left(\beta^x \hat{\beta}^{T+k-n} \frac{1-\alpha^T}{1-\alpha} \frac{\alpha^{T+k+1}}{1-\alpha^T \hat{\beta}^T}\right) & \text{if } N > k \end{cases}$$

Upper bound value Π^*

$$\bar{\Pi}_k = \frac{\alpha(1-\alpha^k)}{1-\alpha} + \left(\alpha^{k+1} \hat{\beta}^k \frac{1-\alpha^T}{1-\alpha} \frac{1}{1-\alpha^T \hat{\beta}^T}\right)$$

Property (Lumpy process requires immediate incorporation)

For a given value of the average time-bucket reduction factor $\hat{\beta}^T$, there exists a limit value $\tilde{\beta}(\hat{\beta}^T)$ given by

$$\tilde{\beta}(\hat{\beta}^T) = \min_{k=1,\ldots,T} \left\{ \alpha^{k+1} \left(\frac{\alpha^{k+1}(1-\alpha^{2T-k-1})}{1-\alpha} + \hat{\beta}^T \frac{1-\alpha^T}{1-\alpha} \frac{\alpha^{2T+1}}{1-\alpha^T \hat{\beta}^T} \right)^{-1} \right\}$$

such that if $\beta < \tilde{\beta}(\hat{\beta}^T)$, then the optimal policy follows:

$$\pi^*(N,x) = 1, \quad \text{for } N = 1,\ldots,T, \; x \in \mathbb{N}^0$$

Table 1
Numerical sensitivity analysis

Case	Indicators	Deterministic $\alpha = 0.99$	Lumpy $\alpha = 0.99$	Deterministic $\alpha = 0.98$	Lumpy $\alpha = 0.98$
	Π^*	85.5694	85.2527	45.6142	45.4551
	$[\Pi^* - \underline{\Pi}]/\Pi^*$	0.82%	0.45%	0.76%	0.42%
$\hat{\beta}^{12} = 0.98$	$\max_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.08%	0.44%	0.12%	0.47%
	$\min_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.05%	0.41%	0.07%	0.41%
	Π^*	75.3188	74.7636	42.6527	42.3526
	$[\Pi^* - \underline{\Pi}]/\Pi^*$	1.63%	0.90%	1.51%	0.81%
$\hat{\beta}^{12} = 0.96$	$\max_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.18%	0.92%	0.26%	0.97%
	$\min_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.10%	0.85%	0.14%	0.85%
	Π^*	60.6923	59.7563	37.7151	37.1598
	$[\Pi^* - \underline{\Pi}]/\Pi^*$	3.22%	1.71%	2.99%	1.40%
$\hat{\beta}^{12} = 0.92$	$\max_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.43%	2.00%	0.59%	2.09%
	$\min_{k=1,\dots,T}[\overline{\Pi}_k - \Pi^*]/\Pi^*$	0.24%	1.81%	0.19%	1.80%

- Cost improvement only known by purchaser (decision maker)
- Company (supplier) defines incentives that reward cost reductions
- Show the existence of better schemes

Rewards regarding lth time bucket:

$$r_l = \sum_{N=1}^{T} w_N \left[c_{(l-1)T+N} - c_{(l-1)T+N-1} \right]$$

w_N : weight given to the Nth period (of time bucket) cost reduction (decision variable)
Purchaser optimal infinite horizon reward

$$R^*_{inc} = \max E \left[\sum_{l=1}^{\infty} \alpha^{lT+1} r_l | c_1 = 1 \right]$$

Rewrite as MDP:
Profit transition

$$\gamma_{inc}(N, x, 0) = 0 \text{ and } \gamma_{inc}(N, x, 1) = w_N(1 - \beta^x)$$

Optimal policy $\pi^*_{inc}(\cdot,\cdot)$

Expected discounted reward for the purchaser

$$V_{inc,\pi^*_{inc}}(j,x) =$$

$$E_{\pi^*_{inc}}\left[\sum_{k=1}^{\infty}\alpha^{t^k}c^k\gamma_{inc}(N^k,x^k,\pi(N^k,x^k))\,\middle|\, t^1=j,c^1=1,x^1=x\right]$$

Local return function

$$h_{inc}(N,x,a,V(\cdot,\cdot)) =$$

$$\begin{cases}\alpha^N\gamma_{inc}(N,x,0)+E_{\Xi}[V(N+1,x+\Xi)], & \text{if } a=0\\ \alpha^N\gamma_{inc}(N,x,1)+\alpha^T\beta^x E_{\Xi^{T-N+1}}[V(1,\Xi^{T-N+1})] & \text{if } a=1\end{cases}$$

Value functions

$$V_{inc,\pi}(N,x) = h_{inc}(N,x,\pi(N,x),V_{inc,\pi}(\cdot,\cdot))$$

$$V_{inc,\pi^*}(N,x) = \min_{a\in\{0,1\}} h_{inc}(N,x,a,V_{inc,\pi^*}(\cdot,\cdot))$$

Lemma (Kth Deterministic change)

Let us consider the admissible policies defined as

$$\underline{\pi}_{inc,k}(N,x) = \begin{cases} 1 \ \textit{if}\ N = k, k = 1, \dots, T \\ 0 \ \textit{otherwise} \end{cases}$$

purchaser reward lower bound

$$V_{\underline{\pi}_{inc,k}}(N,x) = \begin{cases} \alpha^{T+1} w_k \left[1 - \beta^x \hat{\beta}^{k-N} + \frac{\alpha^T \beta^x \beta^{k-N}(1-\hat{\beta}^T)}{1-\alpha^T \hat{\beta}^T} \right] \ \textit{if}\ N \leq k \\ \alpha^{2T+1} w_k \left[1 - \beta^x \hat{\beta}^{k+T-N} + \frac{\alpha^T \beta^x \beta^{k+T-N}(1-\hat{\beta}^T)}{1-\alpha^T \hat{\beta}^T} \right] \ \textit{if}\ N > k \end{cases}$$

The bound for the objective value is

$$\underline{\Pi}_k = \frac{\alpha(1-\alpha^k)}{1-\alpha} + \left(\alpha^{k+1} \hat{\beta}^k \frac{1-\alpha^T}{1-\alpha} \frac{1}{1-\alpha^T \hat{\beta}^T} \right)$$

Incentive Schemes

$$w_{inc1,N} = w_1, N = 1, \ldots, T$$

$$r_{inc1,l} = w_l[c_{lT+1} - c_{(l-1)T+1}]$$

Table 2
Optimal versus first incentive cost performances

β value	Numerical example	Suboptimality indicators	$\alpha = 0.99$	$\alpha = 0.98$
$\hat{\beta}^{12} = 0.98$	Deterministic	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	0.08%	0.12%
$\hat{\beta}^{12} = 0.98$	Lumpy	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	0.44%	0.47%
$\hat{\beta}^{12} = 0.96$	Deterministic	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	0.18%	0.26%
$\hat{\beta}^{12} = 0.96$	Lumpy	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	0.92%	0.97%
$\hat{\beta}^{12} = 0.92$	Deterministic	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	0.43%	0.59%
$\hat{\beta}^{12} = 0.92$	Lumpy	$[\Pi^*_{inc1} - \Pi^*]/\Pi^*$	2.01%	2.09%

$$w_{inc2,N} = \left(1 - \frac{N-1}{T-1}\right) w_1, N = 1, \ldots, T$$

Table 3
Optimal versus second incentive cost performances

β value	Numerical example	Suboptimality indicators	$\alpha = 0.99$	$\alpha = 0.98$
$\hat{\beta}^{12} = 0.98$	Deterministic	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	0.07%	0.07%
$\hat{\beta}^{12} = 0.98$	Lumpy	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	0.36%	0.31%
$\hat{\beta}^{12} = 0.96$	Deterministic	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	0.15%	0.15%
$\hat{\beta}^{12} = 0.96$	Lumpy	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	0.76%	0.64%
$\hat{\beta}^{12} = 0.92$	Deterministic	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	0.37%	0.34%
$\hat{\beta}^{12} = 0.92$	Lumpy	$[\Pi^*_{inc2} - \Pi^*]/\Pi^*$	1.63%	1.36%

Incentives similar to the benefits for the company

$$w_{inc3,N} = \left(1 - \frac{N-1}{T_0 + T - 1}\right) w_1$$

$$w_{inc4,N} = \alpha^{N-1} w_1$$

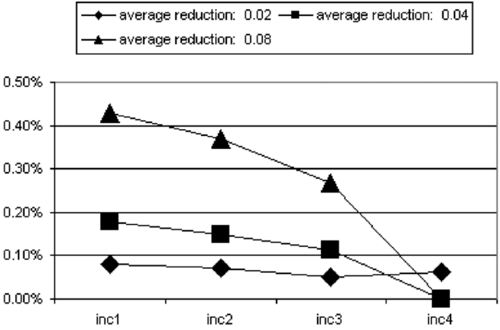


Fig. 2. Incentive performances: suboptimality gaps for the deterministic case ($\alpha = 0.99$).

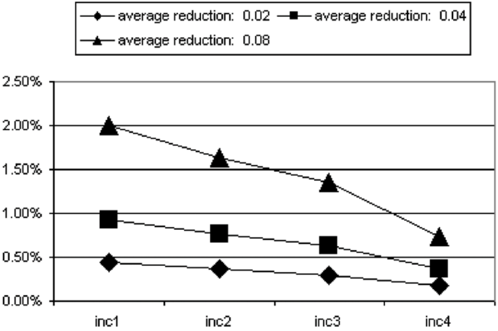


Fig. 3. Incentive performances: suboptimality gaps for the lumpy case ($\alpha = 0.99$).

Conclusion

- Incentive schemes for purchasers
- Common incentive schemes induce suboptimal cost management
- Improvements
- Suboptimality gap: 1) deterministic and periodical cost reductions and 2) lumpy random reductions
- Significant change: more than half of the product cost is material and components

Efficient purchaser incentive when dealing with suppliers implementing continuous improvement plans

Laoucine Kerbache and Christian van Delft

Mateus Hiro Nagata

March 26, 2025